

# Indirect Proofs

# Announcements

- ***Pset 0***
  - Due Monday.
- ***Pset 1***
  - Goes out today, due next **Friday 1pm** (NOT midnight)
  - Start Monday! Schedule will be on Canvas shortly.

# Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

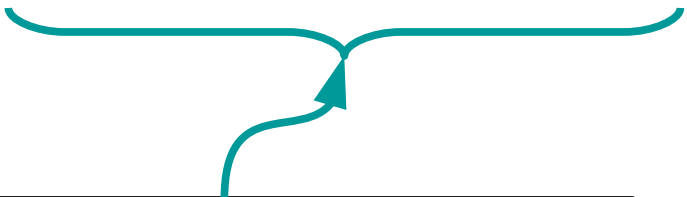
# Logical Implication

If  $n$  is an even integer, then  $n^2$  is an even integer.

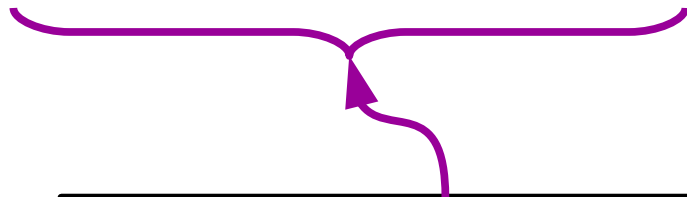
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An ***implication*** is a statement of the form  
“If  $P$  is true, then  $Q$  is true.”

If  $n$  is an even integer, then  $n^2$  is an even integer.



This part of the implication is called the *antecedent*.



This part of the implication is called the *consequent*.

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An ***implication*** is a statement of the form  
“If  $P$  is true, then  $Q$  is true.”

If  $n$  is an even integer, then  $n^2$  is an even integer.

If  $m$  and  $n$  are odd integers, then  $m+n$  is even.

If you prove Cantor's theorem is incorrect,  
then you will receive an A+ in CS103.

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An ***implication*** is a statement of the form  
“If  $P$  is true, then  $Q$  is true.”

# What Implications Mean

**“If there's a rainbow in the sky,  
then it's raining somewhere.”**

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- Implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- Implication says nothing about causality.
  - Rainbows do not cause rain. 😊

# Negations

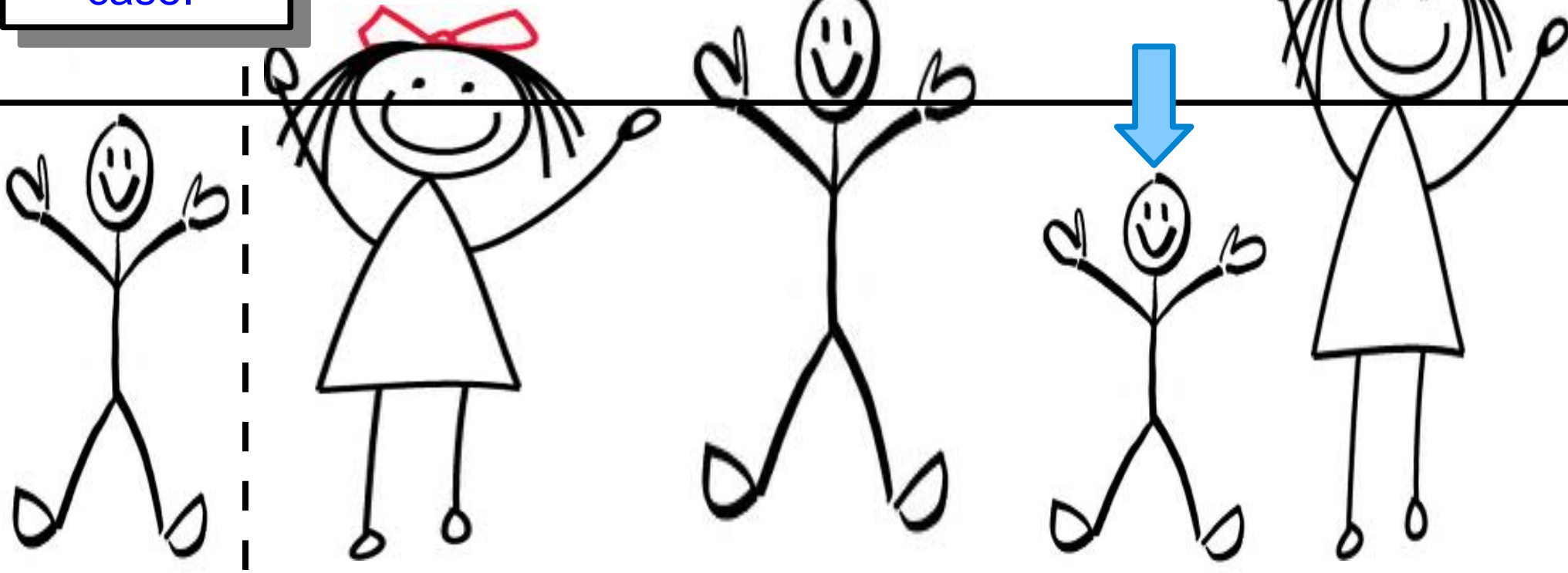
# Negations

- A **proposition** is a statement that is either true or false.
- Some examples:
  - If  $n$  is an even integer, then  $n^2$  is an even integer.
  - $\emptyset = \mathbb{R}$ .
- The **negation** of a proposition  $X$  is a proposition that is true whenever  $X$  is false and is false whenever  $X$  is true.
- For example, consider the proposition “it is snowing outside.”
  - Its negation is “it is not snowing outside.”
  - Its negation is *not* “it is sunny outside.” ⚠

How do you find the negation  
of a *universal* statement?

“All my friends are taller than me”

Statement is  
**false** in this  
case.



Me

My Friends

The negation of the *universal* statement

**Every  $P$  is a  $Q$**

is the *existential* statement

**There is a  $P$  that is not a  $Q$ .**

*(Remember that existential means “at least one.”)*

The negation of the *universal* statement

**For all  $x$ ,  $P(x)$  is true.**

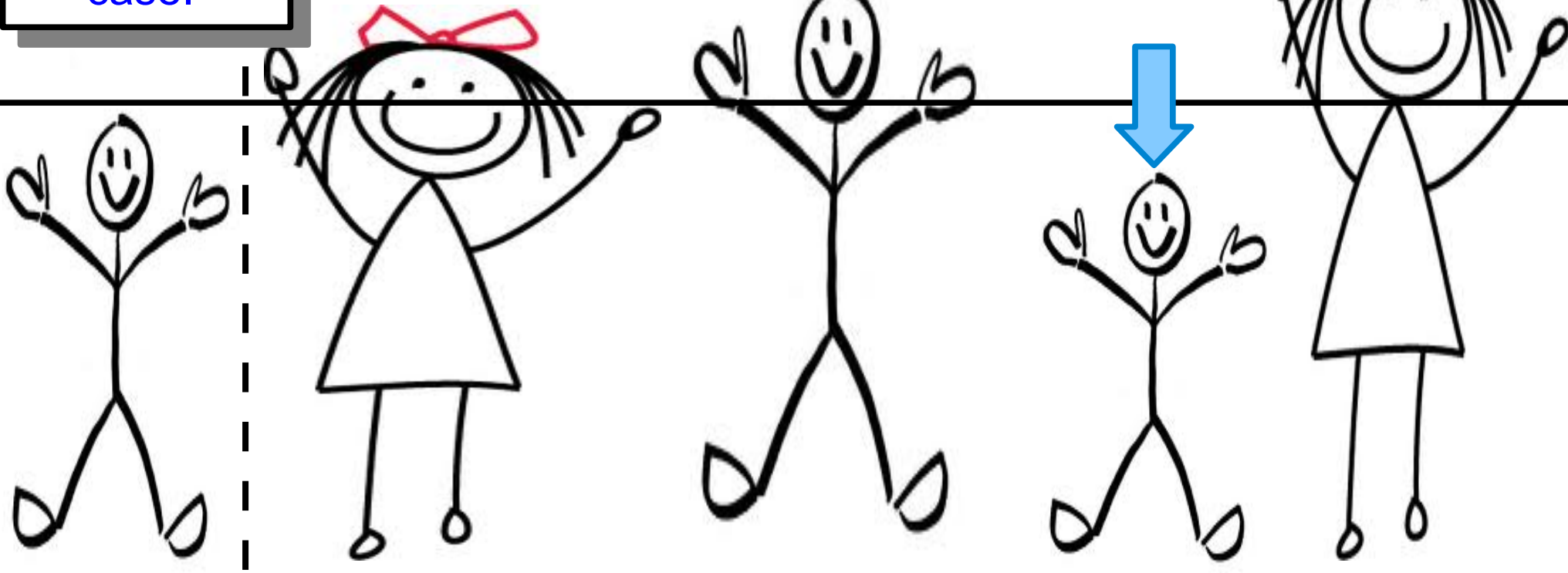
is the *existential* statement

**There exists an  $x$  where  $P(x)$  is false.**

*(Remember that existential means “at least one.”)*

“There exists a friend who is not taller than me”

Negation is **true** in this case.

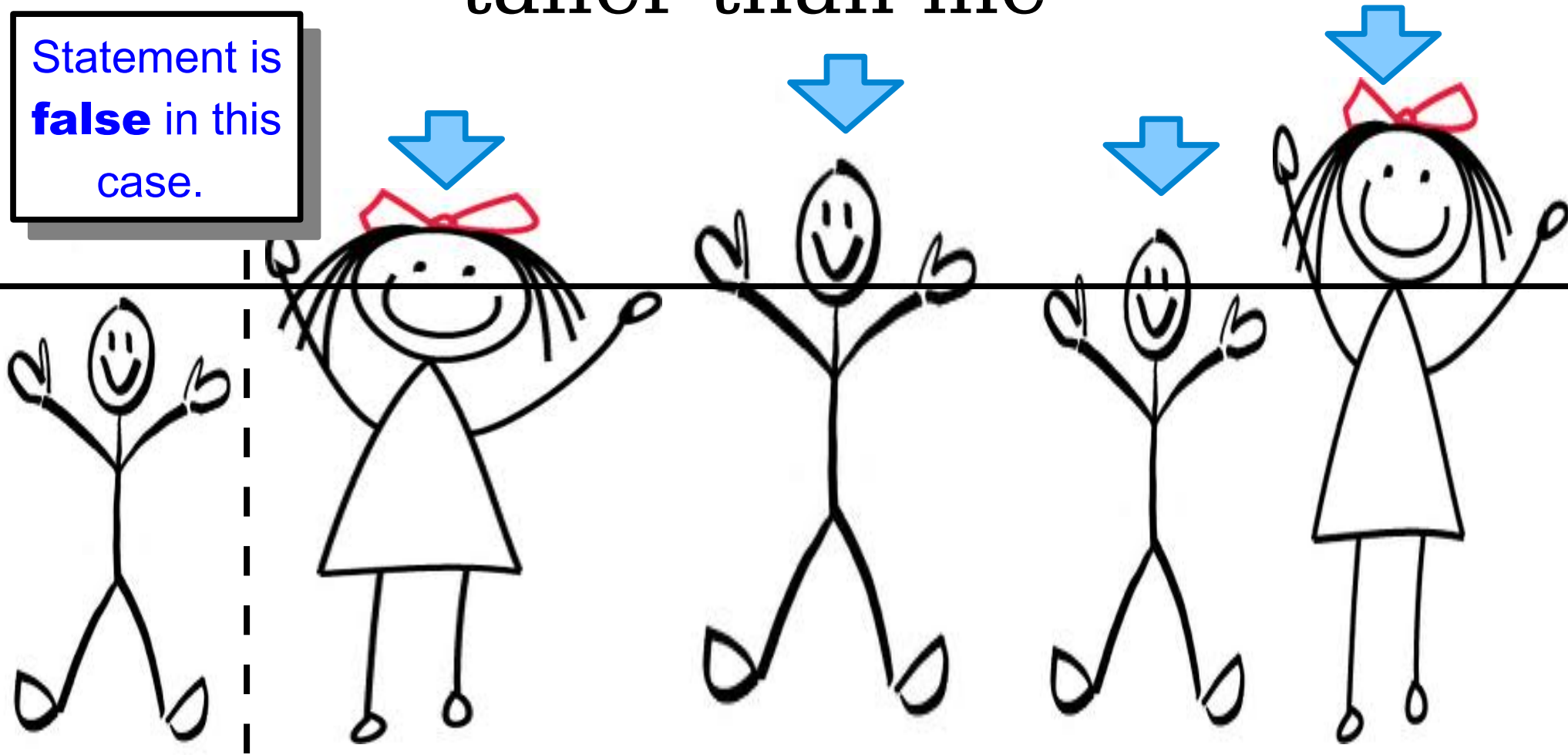


Me

My Friends

“There exists a friend who is not taller than me”

Statement is **false** in this case.



Me

My Friends

The negation of the *existential* statement

**There exists a  $P$  that is a  $Q$**

is the *universal* statement

**Every  $P$  is not a  $Q$ .**

The negation of the *existential* statement

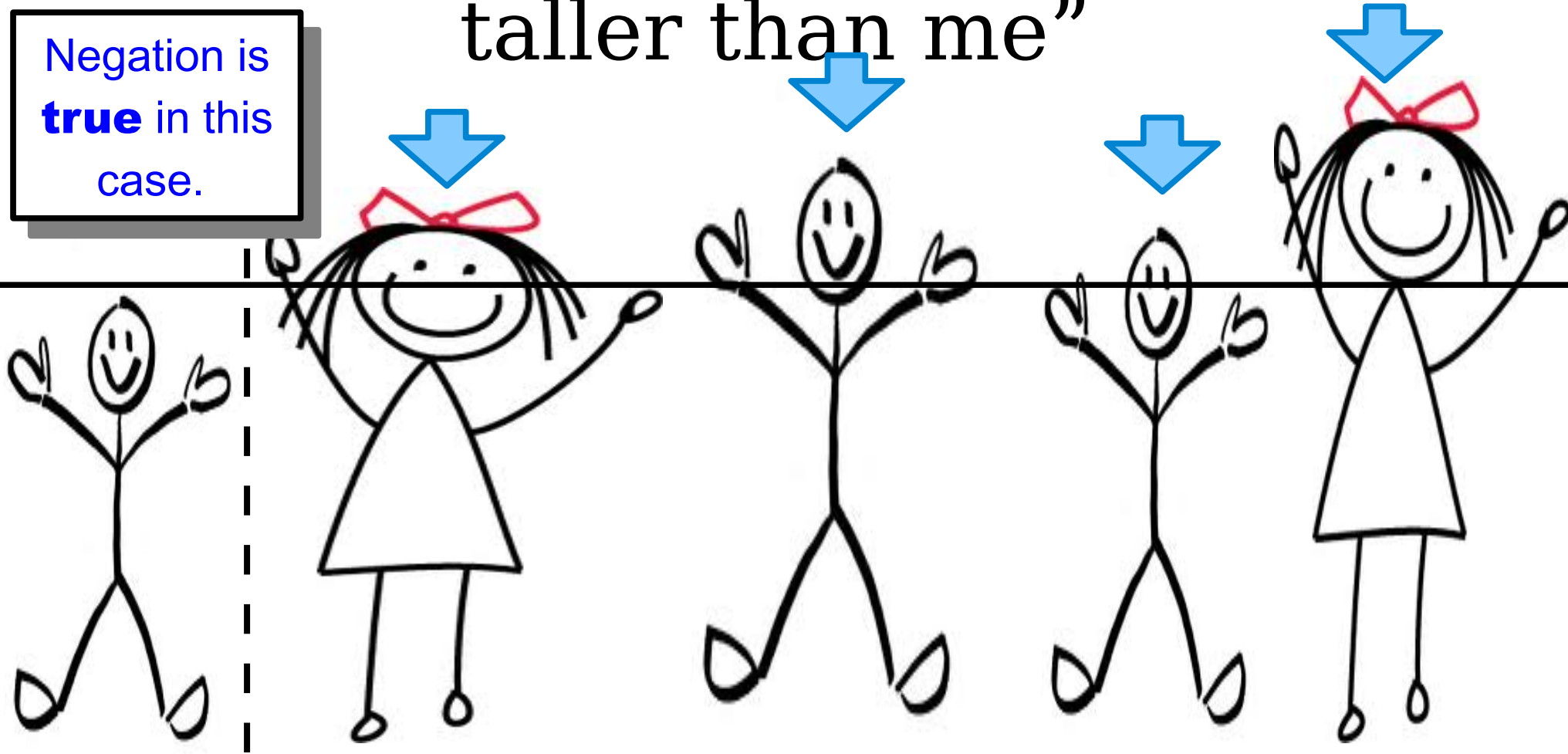
**There exists an  $x$  where  $P(x)$  is true**

is the *universal* statement

**For all  $x$ ,  $P(x)$  is false.**

“All my friends are not not taller than me” → “All my friends are taller than me”

Negation is **true** in this case.



Me

My Friends

How do you negate an implication?

# Negating Implication

Dr. Bailey: “**If** you pick a perfect March Madness bracket, **then** I’ll give you an A+ in CS103.”

Q: under what conditions am I a liar?\*

What if...

- ...you pick a **perfect** bracket and get an A+?
- ...you pick a bad bracket and get an A+?
- ...you pick a **perfect** bracket and get a C?
- ...you pick a bad bracket and get a C?

*\* The way we define negation in logic means these are the conditions under which the negation of my statement is true.*

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# Implication

**If you pick a perfect March bracket, then I'll give you an A+ in CS103."**

Under what conditions am I a liar?\*

- ...you pick a **perfect** bracket and get an A+?
- ...you pick a bad bracket and get an A+?
- ...you pick a **perfect** bracket and get a C?
- ...you pick a bad bracket and get a C?

*\* The way we define negation in logic means these are the conditions under which the negation of my statement is true.*

The negation of the statement

**“For any  $x$ , if  $P(x)$  is true,  
then  $Q(x)$  is true”**

is the statement

**“There is at least one  $x$  where  
 $P(x)$  is true and  $Q(x)$  is false.”**

***The negation of an implication  
is not an implication!***

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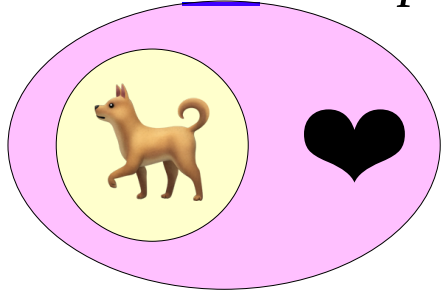
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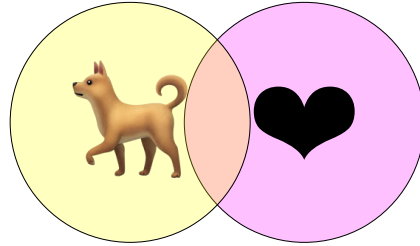
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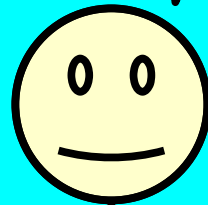
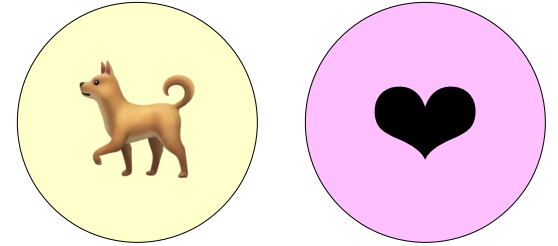
If  $p$  is a puppy,  
then I **do** love  $p$ !



It's  
complicated.



If  $p$  is a puppy,  
then I **don't** love  $p$ !



## *How to Negate Universal Statements:*

**“For all  $x$ ,  $P(x)$  is true”**

becomes

**“There is an  $x$  where  $P(x)$  is false.”**

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## *How to Negate Existential Statements:*

**“There exists an  $x$  where  $P(x)$  is true”**

becomes

**“For all  $x$ ,  $P(x)$  is false.”**

---

Negation of  
“if-then”  
becomes  
“and”!

## *How to Negate Implications:*

**every  $x$ , if  $P(x)$  is true, then  $Q(x)$  is true”**

becomes

**“There is an  $x$  where  $P(x)$  is true and  $Q(x)$  is false.”**

# Proof by Contrapositive

If  $P$  is true, then  $Q$  is true.

If  $Q$  is false, then  $P$  is false.

---

What are the negations of the above two statements?

If  $P$  is true, then  $Q$  is true.

*negates to*



$P$  is true and  $Q$  is false.

If  $Q$  is false, then  $P$  is false.

---

What are the negations of the above two statements?

If  $P$  is true, then  $Q$  is true.

*negates to*



$P$  is true and  $Q$  is false.

*equivalent to*



$Q$  is false and  $P$  is true.

If  $Q$  is false, then  $P$  is false.

---

What are the negations of the above two statements?

If  $P$  is true, then  $Q$  is true.

*negates to*

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*equivalent to*

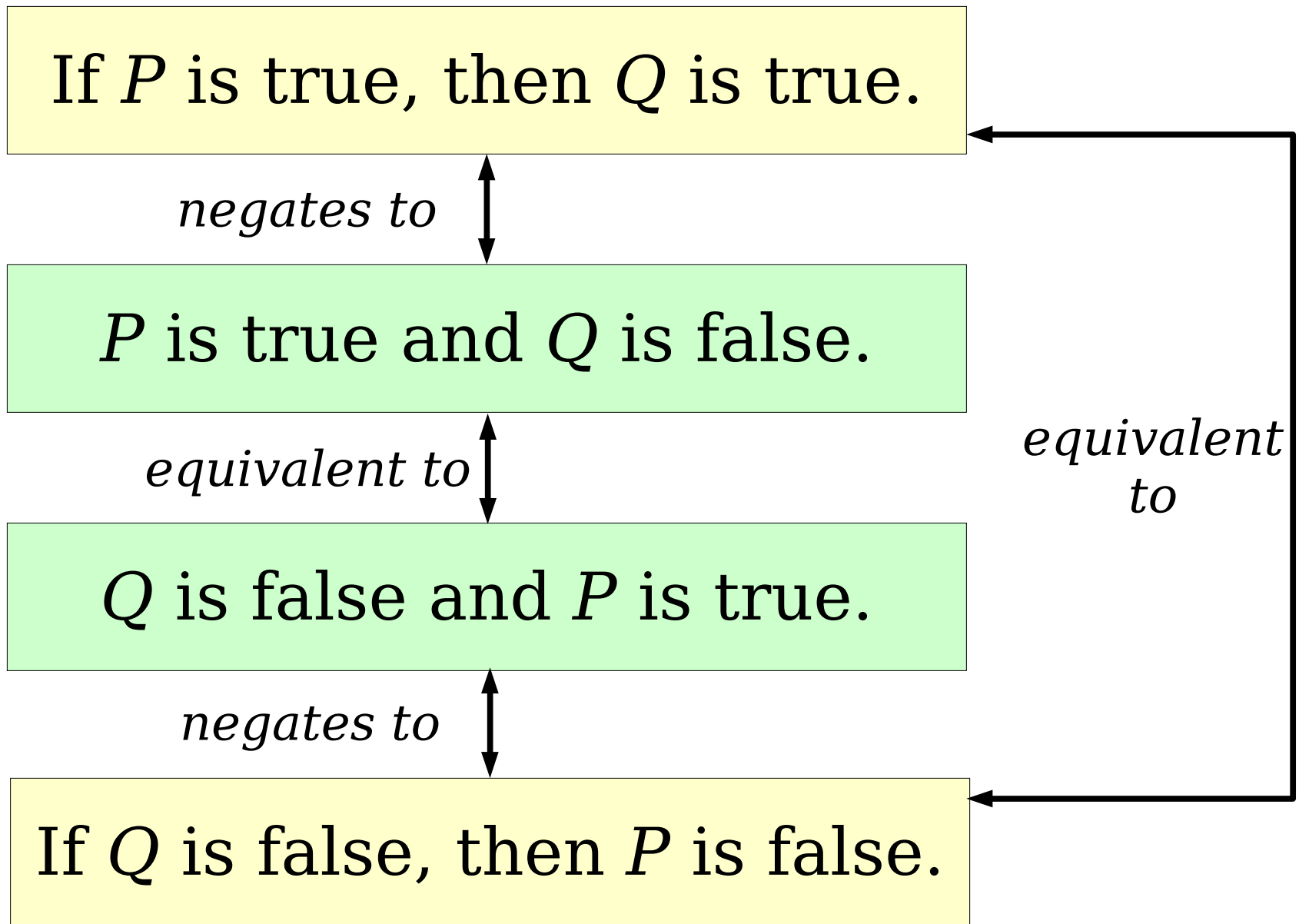
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*negates to*

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What are the negations of the above two statements?



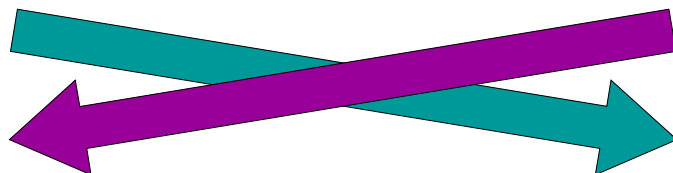
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What are the negations of the above two statements?

# The Contrapositive

- The **contrapositive** of the implication  
If ***P*** is true, then ***Q*** is true  
is the implication  
If ***Q*** is false, then ***P*** is false.
- The contrapositive of an implication means exactly the same thing as the implication itself.

*If it's a puppy, then I love it.*



*If I don't love it, then it's not a puppy.*

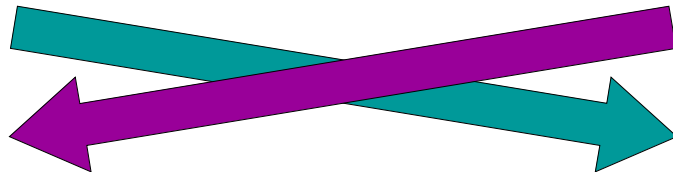
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- The contrapositive of an implication means exactly the same thing as the implication itself.

*If I store cat food inside, then raccoons won't steal it.*



*If raccoons stole the cat food, then I didn't store it inside.*

To prove the statement

“if  $P$  is true, then  $Q$  is true,”

you can choose to instead prove the  
equivalent statement

“if  $Q$  is false, then  $P$  is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

***Theorem:*** For any  $n \in \mathbb{Z}$ , if  $n^2$  is even, then  $n$  is even.

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**Proof:** We will prove the contrapositive of this statement

This is a courtesy to the reader and says  
“heads up! we’re not going to do a  
regular old-fashioned direct proof here.”

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What is the contrapositive of this statement?

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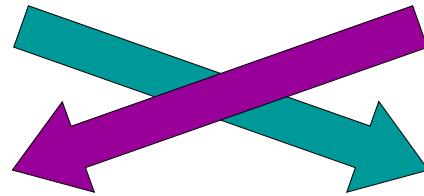
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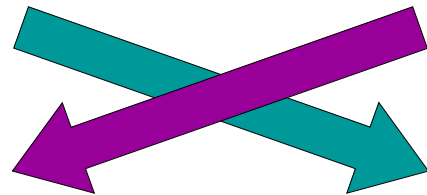


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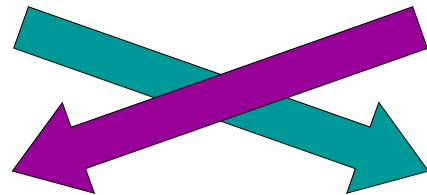
If  $n$  is odd, then  $n^2$  is odd.

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**Proof:** We will prove the contrapositive of this statement, that if  $n$  is odd, then  $n^2$  is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove.

From here, we just do our regular proof template!

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**Proof:** We will prove the contrapositive of this statement, that if  $n$  is odd, then  $n^2$  is odd. Pick an arbitrary odd integer  $n$ . We want to show that  $n^2$  is odd as well.

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We know that  $n$  is odd, which means there is an integer  $k$  such that  $n = 2k + 1$ .

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$$n^2 = (2k + 1)^2$$

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From this, we see that there is an integer  $m$  (namely,  $2k^2 + 2k$ ) such that  $n^2 = 2m + 1$ .

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We know  
integer  
us that

**The general pattern here is the following:**

- 1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.**
- 2. Explicitly state the contrapositive of what we want to prove.**
- 3. Go prove the contrapositive.**

From the  
(namely  
means to  
to show. ■

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# Biconditionals

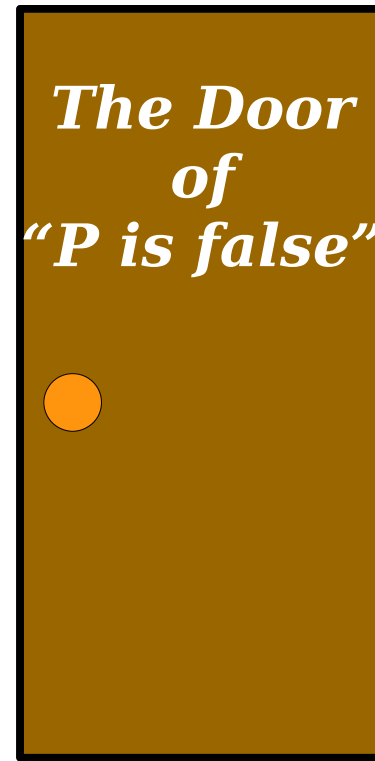
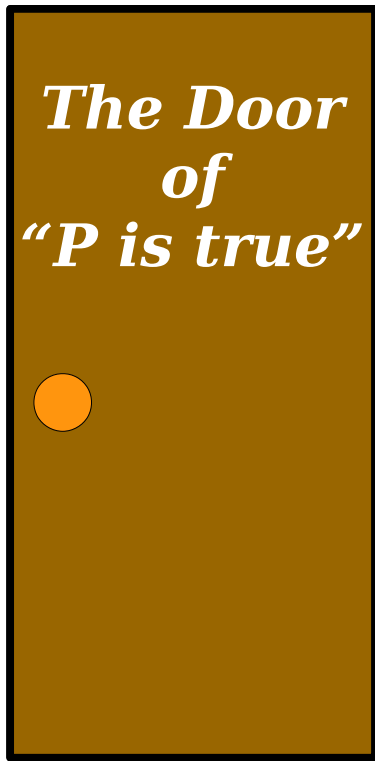
- The previous theorem, combined with what we saw on Wednesday, tells us the following:
  - **For any integer  $n$ , if  $n$  is even, then  $n^2$  is even.**
  - **For any integer  $n$ , if  $n^2$  is even, then  $n$  is even.**
- These are two different implications, one in each direction.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say
  - **for any integer  $n$ :  $n$  is even if and only if  $n^2$  is even.**

# Proving Biconditionals

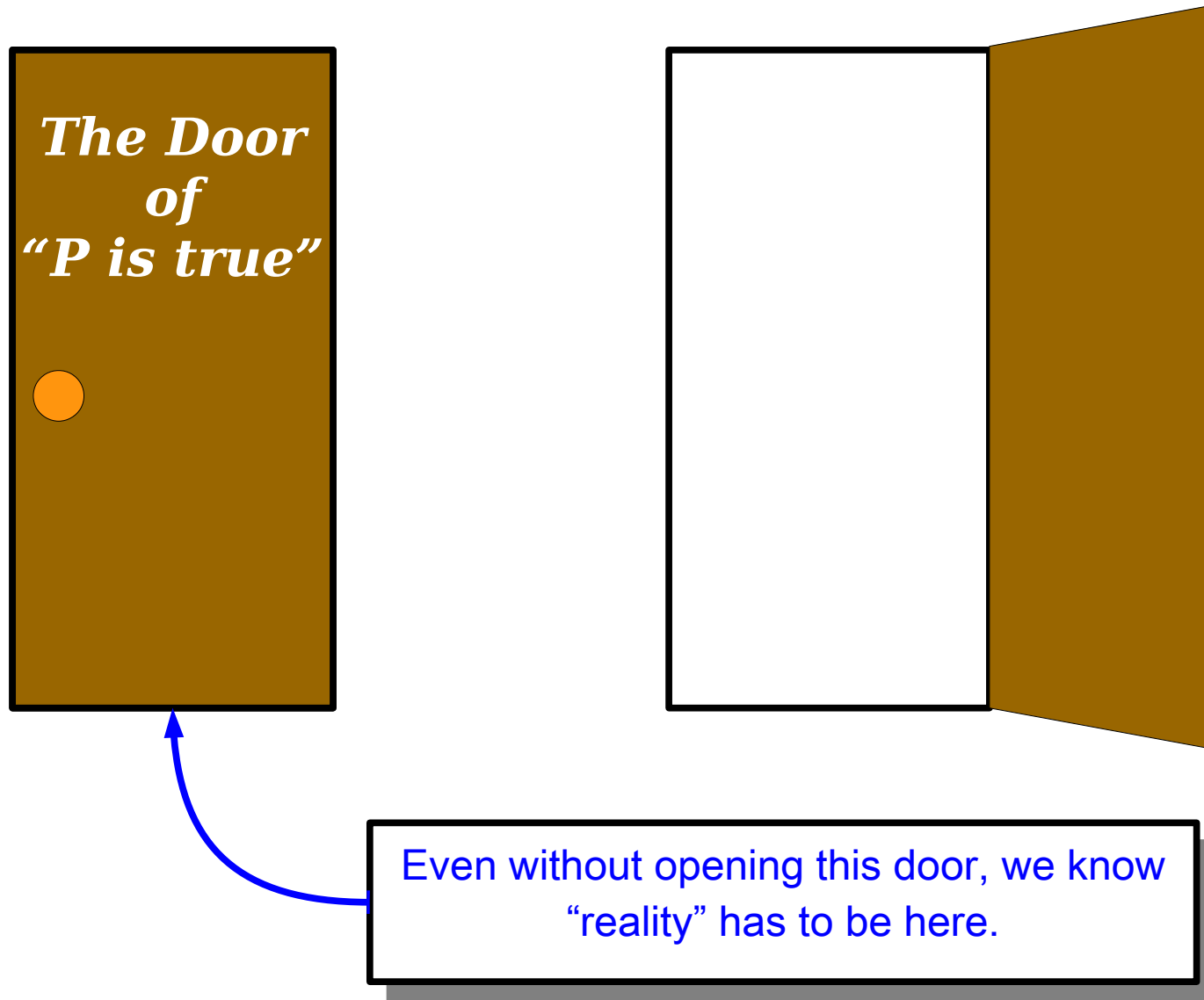
- To prove a theorem of the form  
 **$P$  if and only if  $Q$ ,**  
you need to prove two separate statements.
  - First, that if  $P$  is true, then  $Q$  is true.
  - Second, that if  $Q$  is true, then  $P$  is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

# Proof by Contradiction

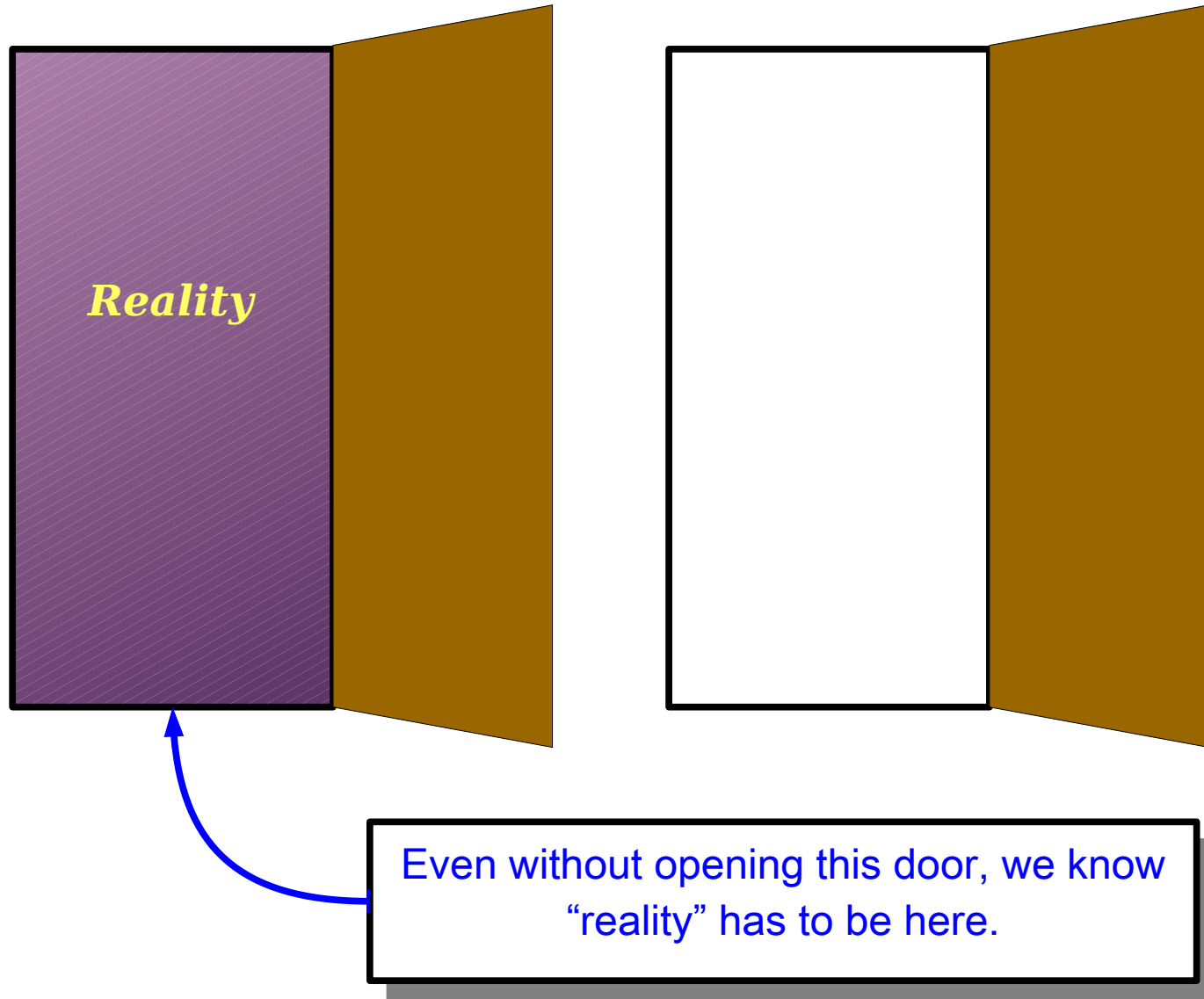
*Every statement in mathematics is either true or false.  
If statement  $P$  is not false, what does that tell you?*



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A ***proof by contradiction*** shows that some statement  $P$  is true by showing that  $P$  isn't false.

# Proof by Contradiction

- **Key Idea:** Prove a statement  $P$  is true by showing that it isn't false.
- First, assume that  $P$  is false. *The goal is to show that this assumption cannot hold.*
- Next, show this assumption leads to an impossible result.
  - For example, we might have that  $1 = 0$ , that  $x \in S$  and  $x \notin S$ , that a number is both even and odd, etc.
- Finally, conclude that since  $P$  can't be false, we know that  $P$  must be true.

An Example: ***Set Cardinalities***

# Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{n \in \mathbb{N} \mid n < 137\}| = 137$
  - $|\mathbb{N}| = \aleph_0$ .
- These span from the finite up through the infinite.
- **Question:** Is there a “largest” set? That is, is there a set that's bigger than every other set?

***Theorem:*** There is no largest set.

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***Proof:***

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***Proof:***

**To prove this statement by contradiction, we're going to assume its negation.**

***Theorem:*** There is no largest set.

***Proof:***

**To prove this statement by contradiction, we're going to assume its negation.**

**What is the negation of the statement  
"there is no largest set?"**

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***Proof:***



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**One option: "there *is* a largest set."**

***Theorem:*** There is no largest set.

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**Notice that we're announcing**

- 1. that this is a proof by contradiction, and**
- 2. what, specifically, we're assuming.**

**This helps the reader understand where we're going.  
Remember – proofs are meant to be read by other  
people!**

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Now, consider the set  $\wp(S)$ . By Cantor's Theorem, we know that  $|S| < |\wp(S)|$ , so  $\wp(S)$  is a larger set than  $S$ .

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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# Proving Implications

- Suppose we want to prove this implication:  
**If  $P$  is true, then  $Q$  is true.**
- We have three options available to us:
  - ***Direct Proof:***
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... what does this look like?

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**What is the negation of our theorem?**

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Negation of  
“if-then”  
becomes  
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(we often use “but” as a  
synonym for “and”)

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**In CS103, please include all these steps in your proofs!**

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then derive a contradiction.

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# What We Learned

- ***What's an implication?***
  - It's a statement of the form "if  $P$ , then  $Q$ ," and states that if  $P$  is true, then  $Q$  is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if  $P$ , then  $Q$ " is "if not  $Q$ , then not  $P$ .")
- ***What's a proof by contradiction?***
  - It's a proof of a statement  $P$  that works by showing that  $P$  cannot be false.

# Your Action Items

- ***Read “Guide to Office Hours,” the “Proofwriting Checklist,” and the “Guide to LaTeX.”***
  - There’s a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we’ll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what’s being asked. That’ll give you time to turn things over in your mind this weekend.

# Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.